COVERING SPACES AND THEIR CLASSIFICATION

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Abstract. In this project we study a classification theorem for covering spaces of a topological space. We first define the fundamental group of $X$ denoted by $\pi_1(X,x_0)$. Then we construct the universal cover for a topological space $X$ and we show that for every subgroup $H$ of $\pi_1(X,x_0)$, there exists a covering space whose fundamental group has the image $H$ under induced homomorphisms of the projection map. Then we prove that this covering space is unique up to covering space isomorphism. In the last part of the classification theorem we show that if we define two different covering space of $X$ with the same topological space having two different base points, when we take the images of the fundamental groups of these two covering spaces, we see that their images are conjugate to each other in $\pi_1(X,x_0)$. Using this, we show that there is a bijection between covering spaces of $X$ and conjugacy classes of subgroups of $\pi_1(X,x_0)$.

Introduction

One of the main purposes in algebraic topology is to find relations between topological structures and algebraic structures. This gives us the opportunity to work on topological problems using the algebraic approach. In this project we give a proof for a classification theorem of covering space of a topological space $X$. This theorem basically says that there is a bijection between covering spaces of $X$ and subgroups of the fundamental group of $X$. For this we first define a correspondence between these two sets, and then we showed this correspondence is surjective and injective up to an isomorphism of covering spaces.

In this project we mainly follow the book of Allen Hatcher which is in bibliography[1]. Therefore, we use mostly the notations of this book and the proof methods we use are similar to this book. In The Section 1, we define some basic concepts of topology and state some results. We finish the same section giving definition of the fundamental group of a topological space and the induced homomorphism of the projection map. Later we observe that the induced homomorphism of the projection map between fundamental groups gives us the relation between covering spaces and subgroups of the fundamental group of the base space.

In The Section 2, we give definition of the covering space and state some important properties of covering spaces. These properties are essential for the covering space theory and we will use these properties frequently in the rest of this report. Then in the same section we show the induced homomorphism indeed gives a correspondence between covering spaces of $X$ and subgroups of fundamental group of $X$. After that point for the rest of this report we prove that this correspondence defines a bijection where $X$ satisfies some topological properties.
To show this correspondence is surjective we first proved for trivial subgroup of fundamental group of base space, there exist a covering space which is related with this subgroup.

In The Section 4, we prove that for an arbitrary subgroup of \( \pi(X, x_0) \), there exists a covering space which corresponds with this subgroup. We directly constructed these covering spaces to prove these two theorems.

In the same section we show that two covering spaces of \( X \) correspond of the same subgroup of \( \pi_1(X, x_0) \) if and only if these two covering spaces are isomorphic with covering space isomorphism. We observe that base point of the covering spaces matters in this uniqueness theorem and if we only consider covering spaces of \( X \) without considering any base point, we can find a bijection between set of covering spaces of \( X \) and conjugacy classes of subgroups of \( \pi_1(X, x_0) \).

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1. Preliminaries and Definitions

In this section we will define some concepts which we will use in this project. Our main aim is to define a group structure using topological concepts which we will call the fundamental group. Since subgroups of the fundamental group of \( X \) will give us the information about covering spaces of \( X \), definition and properties of the fundamental groups are important for this project. We first define some topological concepts.

**Definition 1.1.** Let \( X \) be a topological space. Then a map \( \gamma \) from interval \([0, 1]\) to \( X \) is called a path in \( X \). Also if \( \gamma(0) = \gamma(1) \), then \( \gamma \) is called a loop.

In this definition and for the rest of this report, when we say map, we mean a continuous function. Also we will denote \([0, 1]\) with \( I \). Next we define an equivalence of paths in \( X \) using homotopy. So, we first we define the homotopy of two maps.

**Definition 1.2.** Let \( X \) and \( Y \) be two topological space and \( f \) and \( g \) be two maps from \( X \) to \( Y \). Then a homotopy between \( f \) and \( g \) is a map \( F \) such that \( F : X \times I \longrightarrow Y \) and \( F(x, 0) = f(x) \) and \( F(x, 1) = g(x) \).

A homotopy \( F \) can also be seen as a family of maps from \( X \) to \( Y \) and can be denoted with \( F_t \) where \( F_0 = f \) and \( F_1 = g \). This notation is useful since it simplifies some expressions.

**Remark 1.3.** The relation of being homotopic is an equivalence relation.

If \( f \) is homotopic to \( g \) then it is denoted by \( f \simeq g \) and \([f]\) denotes homotopy class of \( f \). A particular type of homotopy is called path homotopy which is a homotopy where \( F_t(0) \) and \( F_t(1) \) are fixed for all \( t \in I \).

At this point we start to look for whether homotopy classes of paths in a topological space gives a group structure. It turns out it does not always do, but if we set some restrictions to paths we consider then we can obtain a group. First lets define a binary operation of paths in \( X \).
Definition 1.4. Let $X$ be a topological space and $\gamma$, $\eta$ be two paths in $X$ satisfying $\gamma(1) = \eta(0)$ Then $\cdot$ is a binary operation of paths in $X$ defined as follows:

$$\gamma \cdot \eta = \begin{cases} 
\gamma(2t) & \text{if } 0 \leq t \leq 1/2 \\
\eta(2t - 1) & \text{if } 1/2 \leq t \leq 1 
\end{cases}$$

For the rest of the project $\gamma \eta$ will also denote $\gamma \cdot \eta$. Since this binary operation is only used binary operation between paths in this project this notation would not cause any confusion.

This definition of $\cdot$ implies that we can not operate two random paths in $X$. Their starting and ending points matters in this definition. As a consequence, we focus on loops in $X$ so that $\gamma \cdot \gamma$ can be defined. Also as another consequence of the definition is that these loops should start and end at some fixed point so that $\gamma \cdot \eta$ will be defined. So, we set a fixed point in our space. This structure of a topological space and a fixed point is called \textit{pointed space} and denoted with $(X, x_0)$ where $X$ is the topological space and $x_0$ is called base point of $X$. Note that when we are working on a pointed space, unless it is stated otherwise we assume all loops starts from the base point.

In next theorem we use the binary operation defined between paths to define a binary operation between homotopy classes and in fact, this structure is a group.

Theorem 1.5. Let $(X, x_0)$ be a pointed space. Then homotopy classes of loops in $X$, forms a group with binary operation defined as follows:

$$[\gamma] \cdot [\eta] = [\gamma \cdot \eta].$$

Proof. [2]. The first thing we have to prove is that this group operation is well defined. So, let $[\gamma] = [\gamma']$ and $[\eta] = [\eta']$. This implies $\gamma \simeq \gamma'$ and $\eta \simeq \eta'$. Therefore, there exist homotopies $F$ and $F'$ such that $F(t, 0) = \gamma(t)$ and $F(t, 1) = \gamma'(t)$. Also $F'(t, 0) = \eta(t)$ and $F'(t, 1) = \eta'(t)$. Then lets define $\widetilde{F}$ as follows:

$$\widetilde{F}(t, s) = \begin{cases} 
F(2t, s) & \text{if } 0 \leq t \leq 1/2 \\
F'(2t - 1, s) & \text{if } 1/2 \leq t \leq 1 
\end{cases}$$

Notice that $\widetilde{F}(t, 0) = [\gamma \cdot \eta]$ and $\widetilde{F}(t, 1) = [\gamma' \cdot \eta']$. Therefore $[\gamma_1 \cdot \gamma_2] = [\gamma'_1 \cdot \gamma'_2]$ and this shows binary operation is well-defined.

Next we will prove that group axioms are satisfied.

(i) Closeness: If $\gamma$ and $\eta$ loops in $X$ then $\gamma \cdot \eta$ is also a loop in $X$. So $[\gamma \cdot \eta]$ is the homotopy class of a loop in $X$. Therefore this set is closed with defined operation.

(ii) Associativity; Let $f$, $g$ and $k$ be loops in $X$. Then $([f] \cdot [g]) \cdot [k] = [(f \cdot g) \cdot k]$. Lets define

$$F(t, s) = \begin{cases} 
f\left(\frac{4t}{1 + s}\right) & \text{if } 0 \leq t \leq (s + 1)/4 \\
g\left(\frac{4t - 1 - s}{2 - s}\right) & \text{if } (s + 1)/4 \leq t \leq (s + 2)/4 \\
k\left(\frac{4t - s - 2}{2 - s}\right) & \text{if } (s + 2)/4 \leq t \leq 1 
\end{cases}$$

then $F_0 = (f \cdot g) \cdot k$ and $F_1 = f \cdot (g \cdot k)$ by definition of $\cdot$. So $(f \cdot g) \cdot k \simeq f \cdot (g \cdot k)$ so $[(f \cdot g) \cdot k] = [f \cdot (g \cdot k)] = [f] \cdot ([g] \cdot [k])$ So, associativity is satisfied.
Unit element: Unit element of this group is defined as the homotopy class of the constant loop at $x_0$.

Remark 1.6. We will denote homotopy class of constant loop at some point $x \in X$ with $[x]$ for rest of the project.

Let $f$ be a loop in $X$. Then $[f] \cdot [x_0] = [f]$ since $f \cdot x_0 \simeq f$ with the homotopy
$$F(t, s) = \begin{cases} f\left(\frac{2t}{2-s}\right) & \text{if } 0 \leq t \leq s/2 \\ x_0 & \text{if } s/2 \leq t \leq 1 \end{cases}$$

Inverse element; inverse of a $[f]$ is defined as $[\bar{f}]$ where $\bar{f}(t) = f(1-t)$. Let $f$ be a loop in $X$. Then if we define homotopy
$$F(t, s) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq s/2 \\ f(s) & \text{if } s/2 \leq t \leq (2-s)/2 \\ \bar{f}(2t) & \text{if } (2-s)/2 \leq t \leq 1 \end{cases}$$

then $F_0$ is constant loop and $F_1 = f \cdot \bar{f}$. So, $[f] \cdot [\bar{f}] = [x_0]$ and this completes the proof. 

This group is called the fundamental group of $X$ and denoted with $\pi_1(X, x_0)$. Next theorem gives an important property of the fundamental group which will be useful for this report. But before stating this theorem we shall give the definition of a path connected space.

Definition 1.7. A topological space is said to be path connected if for all points $x, y \in X$ there exists $\gamma$ a path in $X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Now we can state our theorem.

Theorem 1.8. If $X$ is a path connected space then for any $x_0, x_1 \in X$, we have
$$\pi_1(X, x_0) \simeq \pi_1(X, x_1).$$

Proof. Since $X$ is path connected there exists a path $\eta$ in $X$ such that $\eta(0) = x_0$ and $\eta(1) = x_1$. Let's define
$$f : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1) \text{ and } \tilde{f} : \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$$
defined as follows:

$$f([\gamma]) = [\bar{\eta}\gamma\eta] \text{ and } \tilde{f} : ([\gamma]) = [\eta\gamma\bar{\eta}].$$

Then $f \tilde{f} = id_{\pi_1(X, x_1)}$ and $\tilde{f} f = id_{\pi_1(x_0)}$.

Also
$$f([\gamma] \cdot [\gamma']) = f([\gamma\gamma']) = [\bar{\eta}\gamma\gamma'\eta] = [\bar{\eta}\gamma\eta\bar{\eta}\gamma'] = [\bar{\eta}\gamma\eta] \cdot [\bar{\eta}\gamma'] = f([\gamma]) \cdot f([\gamma'])$$

Therefore, $f$ is an isomorphism.

When we denote the fundamental group of a path connected space we omit the base point in the notation such as $\pi_1(X)$, since fundamental group does not depend on the base point.
Remark 1.9. The fundamental group is a functor from category of topological spaces to category of groups.

Since it is out of our scope we will not prove that the fundamental group satisfies functor properties. However, the induced homomorphism of fundamental group as a functor is very important for our method of classifying covering spaces. So, we shall give the definition of induced homomorphism of the fundamental group functor.

Definition 1.10. Let $f$ be a map from $X$ to $Y$. Then induced homomorphism $f_*$ of $f$ is a group homomorphism where

$$f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0) \text{ defined by } f_*([\gamma]) = [f\gamma].$$

2. Covering Spaces

We start this chapter with the definition of a covering space of $X$.

Definition 2.1. Let $X$ be a topological space. Then a covering space of $X$ is another topological space $Y$ with a projection map $p : Y \rightarrow X$ satisfying following conditions:

(i) There exists $\{U_i\}_{i \in \Delta}$ an open cover of $X$ such that,

$$p^{-1}(U_i) = \bigsqcup_{j \in \kappa} U^j_i \text{ where } U^j_i \text{ is open in } Y.$$ 

(ii) For every $U_i$, and for every $j$, $p$ is a homeomorphism between $U_i$ and $U^j_i$.

A covering space of $X$ has some important properties. These are usually referred as lifting properties. But, before stating these properties as theorems we need a definition.

Definition 2.2. Let $f : Z \rightarrow X$ be a map and $p : Y \rightarrow X$ be a covering space of $X$. Then lift of $f$ to covering space $Y$, denoted with $\tilde{f}$, is a continuous function such that

$$\tilde{f} : Z \rightarrow Y, \text{ where } p\tilde{f} = f.$$ 

Note that when we say $p : Y \rightarrow X$ is a covering space of $X$, we mean $Y$ is a covering space of $X$ with projection map $p$. We will not give proofs of the following theorems because these theorems are subjects of a general topology lecture rather than our subject that we focus in this project. However, for clearness of theorems we will give references of the proofs in the book which is in the references[1]. Now we can state lifting theorems for covering spaces as follows.

Theorem 2.3. Let $p : Y \rightarrow X$ be a covering space of $X$. Also, let $\gamma$ be a path in $X$ and $\tilde{x}_0 \in p^{-1}(x_0)$ where $\gamma(0) = x_0$. Then there exists unique $\tilde{\gamma}$, lift of $\gamma$ to $Y$ such that $\tilde{\gamma}(0) = \tilde{x}_0$.

Proof. [1, p. 58].

Theorem 2.4. Let $p : Y \rightarrow X$ be a covering space of $X$ and let $F$ be a homotopy of $f$ and $g$ in $X$. Also let $\tilde{f}$ be a lift of $f$ to $Y$. Then there exists a unique $\tilde{F}$ such that $\tilde{F}$ is a lift of $F$ and $\tilde{F}(t, 0) = \tilde{f}$.

Proof. [1, p. 58].
Lifts of maps other than path and homotopy are also important for us to work with covering spaces. So, we will state next two theorems which gives criteria for whether a map has a unique lift to covering space. Also we will use these following theorems in later proofs.

**Theorem 2.5.** Let $f$ be a map from $(Z, z_0)$ to $(X, x_0)$. Also let $p : (Y, y_0) \to (X, x_0)$ be a covering space of $X$. Then $f$ has a unique lift to $Y$ iff $f_*(\pi_1(Z, z_0)) \subset p_*(\pi_1(Y, y_0))$.

**Proof.** Existence part comes from [1, p.59] and uniqueness follows from [1, p.60]. □

Following property of induced homomorphism of the projection map is quite essential for covering space theory. Also, we will frequently use this theorem.

**Theorem 2.6.** Let $p : Y \to X$ be a covering space of $X$. Then

$$p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$$

is always injective.

**Proof.** Assume $[\gamma]$ and $[\gamma']$ elements of $\pi_1(Y, y_0)$ with $p_*([\gamma]) = p_*([\gamma'])$. Then $[p\gamma] = [p\gamma']$ this implies there exist a homotopy between $p\gamma$ and $p\gamma'$. Let $f_t$ be this homotopy. So, $f_0 = p\gamma$ and $f_1 = p\gamma'$. Then we can lift $f_t$ uniquely with $f_0 = \tilde{\gamma}$. Since $f_t$ satisfies $p\tilde{f}_t = f_t$, $p\tilde{f}_1 = f_1$. We know $\tilde{\gamma}'$ also has unique lift starting from base point $y_0$, we conclude $f_1 = \tilde{\gamma}'$. This implies that $\gamma \simeq \gamma'$. So, $p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is injective. □

At this point we have tools to define a relation between covering spaces of a space $X$ and subgroups of $\pi_1(X, x_0)$. So, next theorem gives this relation.

**Theorem 2.7.** Let $(Y, y_0)$ be a covering space of $(X, x_0)$. Then $p_*(\pi(Y, y_0))$ is a subgroup of $\pi_1(X, x_0)$.

**Proof.** If $[\gamma] \in \pi_1(Y, y_0)$ then $\gamma(0) = \gamma(1)$ so $p(\gamma(0)) = p(\gamma(1))$. Thus

$$p_*([\gamma]) \in \pi_1(X, x_0).$$

Also since $p_*$ is injective homomorphism, image of $\pi_1(Y, y_0)$ is also a group. Therefore $p_*(\pi(Y, y_0)) \leq \pi_1(X, x_0)$. □

This relation between covering spaces of a space $X$ and subgroups of $\pi_1(X, x_0)$ is the main tool for our proof of the classification theorem of covering spaces. We will consider this relation as a function from set of covering spaces of $X$ to the set of subgroups of $\pi_1(X, x_0)$ and we will state it as follow:

$$\phi : \{\text{pointed covering spaces of } X\} \to \{\text{subgroups of } \pi_1(X, x_0)\}$$

$$\phi((Y, y_0)) = p_*(\pi(Y, y_0))$$

where $p : (Y, y_0) \to (X, x_0)$ is a covering space of $X$. For the rest of the report we will prove that this function is surjective and injective. To prove this we will answer following questions:

(i) For any subspace $H$ of $\pi_1(X, x_0)$ is there a covering space of $X$ we denote with $(Y, y_0)$ and satisfies, $p_*(\pi(Y, y_0)) = H$?

(ii) Is this covering space is unique up to an isomorphism?
For the first question, first we will give answer for trivial subgroup of $\pi_1(X, x_0)$. But first notice that if for a covering space $Y$, $p_*(\pi(Y, y_0))$ is trivial, then $\pi_1(Y, y_0)$ is trivial since $p_*$ is injective.

**Definition 2.8.** A topological space with trivial fundamental group is called simply connected and simply connected covering space of a topological space $X$ is called the universal cover of $X$.

In the next section we will investigate the existence of the universal cover for a topological space.

### 3. The Existence of Universal Cover

Before giving the theorem about the existence of the universal cover of $X$, we will first point out a necessary property, which has the following definition, for $X$ in order to have the universal cover.

**Definition 3.1.** A topological space is said to be semi-locally simply connected if it satisfies that for all $x \in X$ there exists $U$ open subset of $X$ such that $x \in U$ and

$$i_*: \pi_1(U, x) \rightarrow \pi_1(X, x)$$

is trivial map where $i: U \hookrightarrow X$ is inclusion map of $U$ into $X$.

**Theorem 3.2.** Let $X$ be a topological space. If $X$ has a universal cover, then $X$ is semi-locally simply connected.

*Proof.* Assume $X$ has a simply connected covering space $\tilde{X}$. Then for all $x \in X$ there exists $U \subset X$ such that $x \in U$ and $p$ maps $U$ into $\tilde{U} \subset \tilde{X}$ homeomorphically. Let $\gamma$ be a loop in $U$ starting at $x$ and let $\tilde{x} \in p^{-1}(x)$. Then $\gamma$ has a unique lift $\tilde{\gamma}$ in $\tilde{U}$ starting from $\tilde{x}$. $i(\gamma) = p(\tilde{\gamma})$. Since $\tilde{X}$ is simply connected $[\tilde{\gamma}]$ is trivial. So, $p_*(\tilde{\gamma})$ and $i_*([\gamma])$ is trivial. □

Therefore, we need to add this condition to our set-up when we state our theorem for existence of universal cover. Also we will add two more conditions to our theorem. These conditions are required since our method of classifying covering spaces are using paths and the group of their homotopies. Also we will need these properties while proving further theorems. One of them is path connectedness which we have already defined. The other property is called locally path connectedness which is defined as follows.

**Definition 3.3.** A topological space $X$ is said to be locally path connected if for every $x \in X$ and for any neighbourhood $U$ of $x$ in $X$, there exists an open path connected subset $V$ of $U$ such that, $x \in V \subset U$.

So, the theorem is as follows.

**Theorem 3.4.** Let $(X, x_0)$ be a pointed space and let $X$ be path connected, locally path connected, semi-locally simply connected. Then there exists $(Y, y_0)$ a locally path connected and path connected covering space of $(X, x_0)$ such that $p_*(\pi_1(Y, y_0)) = id$.

For the rest of this section we will prove this theorem constructing a simply connected covering space of $X$. If $(\tilde{X}, \tilde{x}_0)$ is simply connected, then $\pi_1(\tilde{X}, \tilde{x}_0)$ is trivial. Thus $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ will be trivial.
First we define
\[ \widetilde{X} = \{ [\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0 \} \]
where \([\gamma]\) denotes path homotopy class of \(\gamma\). Next we define projection map
\[ p : \widetilde{X} \rightarrow X \quad [\gamma] \mapsto \gamma(1). \]
Since path homotopy fixes endpoints this map is well defined.

Before we define topology of \(\widetilde{X}\) we will first set a particular open cover of \(X\) and construct topology of \(\widetilde{X}\) as the topology which basically satisfies covering space properties with this open cover. So, we first define
\[ \mathcal{U} = \{ U \mid \text{ is a path connected open set in } X \text{ such that } i_*(\pi_1(U)) \text{ is trivial} \} \]
where \(i : U \rightarrow X\) is inclusion map.

Note that we omitted base point of \(U\) denoting \(\pi_1(U, x)\) since \(U\) is path connected.

**Theorem 3.5.** \(\mathcal{U}\) is a basis for topology of \(X\).

**Proof.** Since \(X\) is semi-locally simply connected for all \(x \in X\) there exists \(U\) such that \(x \in U\) and \(i_*(\pi_1(U))\) is trivial. Every \(x\) in \(X\) is element of a \(U \in \mathcal{U}\). Also let \(x \in U \cap U'\). We know \(X\) is locally path connected so there exists \(W \in U \cap U'\) where \(W\) is an open path connected subset of \(U \cap U'\). Then
\[ \pi_1(W) \xrightarrow{i_*} \pi_1(U) \xrightarrow{i_*} \pi_1(X). \]
So \(W \in \mathcal{U}\). Therefore \(\mathcal{U}\) is a topological basis of \(X\). \(\square\)

Now we start to construct a topology for \(\widetilde{X}\). So, we first define
\[ U_{[\gamma]} = \{ [\gamma\eta] \mid \eta \text{ is a path in } X \text{ where } \eta(0) = \gamma(1) \} \]
and
\[ \widetilde{\mathcal{U}} = \{ U_{[\eta]} \mid U_{[\gamma]} \text{ as defined above} \}. \]

**Theorem 3.6.** \(\widetilde{\mathcal{U}}\) satisfies conditions to be a topological basis.

Before proving this theorem we will first prove a lemma to use further in proof.

**Lemma 3.7.** \([\gamma] \in U_{[\gamma']} \iff U_{[\gamma]} = U_{[\gamma']}\).

**Proof.** If \([\gamma] \in U_{[\gamma']}\) then \([\gamma] = [\gamma'\eta]\). Let \([\tau] \in U_{[\gamma]}\) this implies that \([\tau] = [\gamma\eta']\) therefore \([\tau] = [\gamma'\eta']\) since \(\eta\) and \(\eta'\) are in \(U\) \(\eta\eta'\) is in \(U\). So \([\tau] \in U_{[\gamma']}, U_{[\eta]} \subset U_{[\gamma']}\). If \([\tau'] \in U_{[\gamma]}\) then \([\tau'] = [\gamma'\eta'] = [\gamma\eta'\eta]\). \(\eta\) and \(\eta'\) are in \(U\), so \([\tau'] \in U_{[\gamma']}\). Conversely if \(U_{[\gamma]} = U_{[\gamma']}\), then since \([\gamma] \in U_{[\eta]}, [\gamma] \in U_{[\gamma']}\). \(\square\)

**Proof of Theorem 3.6.** For all \([\gamma] \in \widetilde{X}, [\gamma] \in U_{[\gamma]}\). Let \(U_{[\gamma]}\) and \(V_{[\gamma']}\) are elements of \(\widetilde{\mathcal{U}}\), since \(U\) and \(V\) are elements of \(\mathcal{U}\), which is a topological basis, then there exists \(W \subset U \cap V\) and \(W \in \mathcal{U}\). Then we observe that if \([\gamma''] \in U_{[\gamma']} \cap V_{[\gamma'']}\) so \(U_{[\gamma'']} = U_{[\gamma]}\) and \(V_{[\gamma'']} = V_{[\gamma']}\) by the previous lemma. Thus \([\gamma''] \in U_{[\gamma'']} \cap V_{[\gamma'']}\). Then \(W_{[\gamma'']} \subset U_{[\gamma'']} \cap V_{[\gamma'']}\). So, \(\widetilde{\mathcal{U}}\) defines a topological basis for \(\widetilde{X}\). \(\square\)
We define the topology of $\tilde{X}$ as the topology generated by basis $\tilde{U}$. As the last step of construction we set $[x_0]$ as the base point of $\tilde{X}$, where $[x_0]$ denotes path homotopy class of constant path at $x_0$. With this we have finished our construction of simply connected covering space of $X$ and in the rest of this section we will prove that with this construction $(\tilde{X}, [x_0])$ is path connected and simply connected covering space of $(X, x_0)$.

**Theorem 3.8.** $(\tilde{X}, [x_0])$ is a covering space of $X$ with $p : (\tilde{X}, [x_0]) \rightarrow (X, x_0)$ where $p([\gamma]) = \gamma(1)$.

**Proof.** Set $U$ as before $X$. We know $U$ is an open cover of $X$. $p^{-1}(U) = \bigcup U_{[\gamma]}$ where $\gamma(1) \in U$. Let for some $\gamma$ and $\gamma'$, $U_{[\gamma]} \cap U_{[\gamma']} \neq \emptyset$. Then there exists $[\gamma''] \in U_{[\gamma]}$ and $[\gamma''] \in U_{[\gamma']}$. Then by previous lemma $U_{[\gamma'']} = U_{[\gamma]} = U_{[\gamma']}$. Therefore, $\bigcup U_{[\gamma]}$ is a disjoint union. Next we need to show $p : U_{[\gamma]} \rightarrow U$ is a homeomorphism.

Since $U$ is path connected for all $x \in U$ there exists $\eta$ such that $\gamma(0) = \gamma(1)$ and $\eta(1) = x$ and by definition $p([\gamma\eta]) = x$ and since $[\gamma\eta] \in U_{[\gamma]}$ $p : U_{[\gamma]} \rightarrow U$ is surjective. Since $\pi_1(U) \hookrightarrow \pi_1(X)$ is trivial $p : U_{[\gamma]} \rightarrow U$ is injective. To show that let $[\gamma\eta]$ and $[\gamma\eta']$ be two elements of $U_{[\gamma]}$ such that $p([\gamma\eta]) = p([\gamma\eta'])$. Since $\pi_1(U) \hookrightarrow \pi_1(X)$ is trivial and $\eta$, $\eta'$ are in $U$ and their endpoints are equal to each other, $[\eta] = [\eta']$ in $X$.

Therefore, $[\gamma\eta] = [\gamma\eta']$ in $X$, so $p : U_{[\gamma]} \rightarrow U$ is injective.

To show $p : \tilde{X} \rightarrow X$ is continuous let $V \in U$. Then $p^{-1}(V)$ is union of $V_{[\gamma]}$ where $\tau(1) \in V$. Since we know $V_{[\gamma]} \in \tilde{U}$ by definition of $\tilde{U}$, $p^{-1}(V)$ is open.

Conversely, let $V_{[\gamma]} \in \tilde{U}$, then $p(V_{[\gamma]}) = V \in U$, therefore $p(V)$ is open. We conclude that $p$ and $p^{-1}$ are both continuous. So, $p : U_{[\gamma]} \rightarrow U$ is a homeomorphism. \(\Box\)

Since covering spaces of $X$ is locally homeomorphic to $X$, locally path connectedness is directly inherited by its covering spaces. As the last part of the proof we will prove next theorem.

**Theorem 3.9.** $\tilde{X}$ is path connected and simply connected.

**Proof.** Let $[\gamma] \in \tilde{X}$. Then we define a path in $\tilde{X}$ as follows:

$$f : I \rightarrow \tilde{X}, \quad t \mapsto [h_t]$$

where $h_t(s) = \gamma(ts)$. In this definition $h_t$ is basically a path from $x_0$ to $\gamma(t)$. Therefore, $f(0) = [x_0]$ and $f(1) = [\gamma]$. So, $f$ is a path from $[x_0]$ to $[\gamma]$. Therefore $\tilde{X}$ is path connected. So, it remains to show $\tilde{X}$ is simply connected to finish the proof.

Let $\tilde{\gamma}$ be a loop in $\tilde{X}$. Then $p\tilde{\gamma} = \gamma$ is a loop in $X$. Lets define a lift of $\gamma$ as follows.

$$f : I \rightarrow X, \quad t \mapsto [h_t]$$

where $h_t$ is defined as a path from $x_0$ to $\gamma(t)$. Then, $f$ is a path in $\tilde{X}$ and $pf = \gamma$ since

$$pf(s) = p([h_t]) = h_t(1) = \gamma(t).$$

So, by unique lifting property of covering spaces $\tilde{\gamma} = f$. Since $\tilde{\gamma}$ is a loop $\tilde{\gamma}(0) = \tilde{\gamma}(1)$, so, $f(0) = f(1)$. By definition of $f$ we have

$$[x_0] = f(0) = f(1) = [\gamma].$$
Therefore, $\gamma$ is homotopic to the constant path. Since the induced homomorphism of the projection map is injective, $\tilde{\gamma}$ is homotopic to trivial path in $\tilde{X}$. Since we started with an arbitrary loop in $\pi_1(\tilde{X}, [x_0])$, we conclude that $(\tilde{X}, [x_0])$ is simply connected.

Therefore, we have finished the construction of the path connected, locally path connected and simply connected covering space of a locally path connected, path connected and semi-locally simply connected space $X$.

4. Classification of Covering Spaces

In previous section we showed that if $X$ is path connected, locally path connected and semi-locally simply connected then there exists a covering space of $X$ satisfying $p_*(\pi_1(X, x_0))$ is trivial subgroup of $\pi_1(X, x_0)$. In this chapter we will continue to show this correspondence between covering spaces $X$ and subgroups of $\pi_1(X, x_0)$ defined with induced homomorphism is surjective. So, next theorem says that what we have proved for trivial subgroup $\pi_1(X, x_0)$ is also true for any subgroup of $\pi_1(X, x_0)$

**Theorem 4.1.** Let $(X, x_0)$ be a path connected, locally path connected and semi-locally simply connected space and let $H$ be any subgroup of $\pi_1(X, x_0)$. Then there exists a path connected covering space of $X$ such that:

$$p_*(\pi_1(Y, y_0)) = H \leq \pi_1(X, x_0).$$

**Proof.** Assume $H \leq \pi_1(X, x_0)$. Then let $(\tilde{X}, \tilde{x}_0)$ be the simply connected covering space of $X$ that we defined in the proof of Theorem 2.12. Then we define a relation between elements of $\tilde{X}$ stated as follows:

$$[\gamma] \sim [\gamma'] \text{ if } \gamma(1) = \gamma'(1) \text{ and } [\gamma \tilde{\gamma}'] \in H.$$  

First we will prove that $\sim$ is an equivalence relation.

(i) $\gamma \sim \gamma$ since $\gamma(1) = \gamma(1)$ and $[\gamma \tilde{\gamma}] = [x_0]$. $H$ is a subgroup so it contains identity element. Therefore, $[\gamma \tilde{\gamma}] = [x_0] \in H$.

(ii) If $\gamma \sim \gamma'$ then $\gamma(1) = \gamma'(1)$ and $[\gamma \tilde{\gamma}'] \in H$. Since $H$ is subgroup inverse of $[\gamma \tilde{\gamma}]$ is $H$. Therefore, $[\gamma \tilde{\gamma}'] = [\gamma' \tilde{\gamma}] \in H$. So, $\gamma' \sim \gamma$.

(iii) If $\gamma \sim \gamma'$ and $\gamma' \sim \gamma''$ then $\gamma(1) = \gamma'(1) = \gamma''(1)$. Also, $[\gamma \tilde{\gamma}'] \in H$ and $[\gamma' \tilde{\gamma}''] \in H$. Since $H$ is subgroup it is closed, so $[\gamma \gamma'] \cdot [\gamma' \tilde{\gamma}'] = [\gamma \gamma''] \in H$.

This proves $\sim$ is equivalence relation. Then we can define quotient space with quotient topology as follows: $\tilde{X}/\sim$ with quotient topology. We set base point of this quotient space as equivalence class of $[x_0]$ with respect to $\sim$. For the rest of this proof we will denote this space with $(Y, y_0)$. We will show $Y$ satisfies covering space properties as a consequence of next lemma.

**Lemma 4.2.** Let $\tilde{X}$ and $\sim$ is defined as above. Then for any point $[\tau] \in U_{[\gamma]}$ and $[\tau'] \in U_{[\gamma']}$, if $[\tau] \sim [\tau']$ then for all $\eta \in U$, $[\tau \eta] \sim [\tau' \eta]$.

**Proof.** Let $[\tau] \sim [\tau']$ for $[\tau] \in U_{[\gamma]} = U_{[\gamma']}$ and $[\tau'] \in U_{[\gamma']} = U_{[\gamma']}$. Then by definition of equivalence relation $\tau(1) = \tau'(1)$ and $[\tau \tilde{\tau}'] \in H$. Then clearly $\tau \eta(1) = \tau' \eta(1)$. Also

$$[\tau \tilde{\tau}'] = [\tau \eta \tilde{\tau}'] = [\tau \eta(\tau' \eta)].$$

Therefore, $\tau \eta \sim \tau' \eta$. 

□
Therefore, any two $U_{[\gamma]}$ and $U_{[\gamma']}$ are equivalent to each other or completely disjoint under $\sim$ relation. Therefore, $p^{-1}(U)$ is disjoint union of $U_{[\gamma]}$'s. Also since we know $p : U_{[\gamma]} \longrightarrow U$ is a homeomorphism, $Y$ is a covering space of $X$. Now it remains to prove that $p_*(\pi_1(Y, y_0)) = H$.

To show that let $[\gamma] \in p_*(\pi_1(Y, y_0))$. Then we define $[\gamma]$ as we did in the preceding section. So, $\gamma_t$ is the path from $x_0$ to $\gamma(t)$. So, $[\gamma]$ is a lift to $Y$. Since $[\gamma] \in p_*(\pi_1(Y, y_0))$ there exists a path in $\tilde{\gamma}$ such that $[\tilde{\gamma}] \in \pi_1(Y, y_0)$ and $p\tilde{\gamma} = \gamma$. From unique lifting property we know $\tilde{\gamma} = \gamma_t$. Since $\tilde{\gamma}$ is a loop, $y_0 = [\gamma_0] = [\gamma_1] = [\gamma]$. Therefore, $[\gamma] \sim [x_0]$ and $[\gamma] \in H$.

Conversely, let $[\gamma] \in H$. Then define its lift as $[\gamma]$ as above. Then since $[\gamma] \sim [x_0]$, $[x_0] = [\gamma_0] = [\gamma_1] = [\gamma]$, so $[\gamma] \in p_*(\pi_1(Y, y_0))$ and this completes the proof. □

At this point we showed our correspondence function is surjective. Next thing to show is it is injective. However, our classification of covering space will be unique up to an isomorphism between covering space. Therefore before giving the theorem of uniqueness part of this correspondence, we first give definition of covering space isomorphism.

**Definition 4.3.** If $(Y_1, y_1)$ and $(Y_2, y_2)$ are two covering spaces of $(X, x_0)$ with corresponding projection maps as follows:

$$p_1 : (Y_1, y_1) \longrightarrow (X, x_0) \text{ and } p_2 : (Y_2, y_2) \longrightarrow (X, x_0).$$

Then an isomorphism of covering spaces between $Y_1$ and $Y_2$ is a homeomorphism

$$f : (Y_1, y_1) \longrightarrow (Y_2, y_2) \text{ satisfying } p_1 = p_2 f.$$

Two covering spaces of $(X, x_0)$ are said to be isomorphic if there exist an covering space isomorphism between them.

**Theorem 4.4.** Let $X$ be path connected, locally path connected and semi-locally simply connected space. Also let $(Y_1, y_1)$ and $(Y_2, y_2)$ be two path connected covering spaces of $(X, x_0)$. Then $(Y_1, y_1)$ and $(Y_2, y_2)$ are isomorphic iff

$$(p_1)_*(\pi_1(Y_1, y_1)) = (p_2)_*(\pi_1(Y_2, y_2)).$$

Before proving this theorem we should note that base point of covering space $Y$ matters in this theorem. We mean by this, $p_1_*(\pi_1(Y, y_1)) \neq p_2_*(\pi_1(Y, y_2))$ in general.

**Proof.** Assume $(Y_1, y_1)$ and $(Y_2, y_2)$ are isomorphic to each other. Then there exists $f$ such that $f : (Y_1, y_1) \longrightarrow (Y_2, y_2)$ satisfying $p_1 = p_2 f$ a homeomorphism where $p_1 = p_2 f$. Then let $[\gamma] \in p_1_*(\pi_1(Y_1, y_1))$. This implies $[\gamma] \in \pi_1(Y_1, y_1)$ such that $p_1_*(\gamma) = [\gamma]$, so $p_1(\gamma) = \gamma$. Since $f$ is covering isomorphism $p_2 f(\gamma) = \gamma$. Since $f$ is continuous $f(\gamma)$ is a loop in $(Y_2, y_2)$. Therefore, $[\gamma] \in \pi_1(Y_2, y_2)$ and $p_2(\gamma) = \gamma$. So, $[\gamma] \in p_2_*(\pi_1(Y_2, y_2))$.

Conversely, assume $p_1_*(\pi_1(Y_1, y_1)) = p_2_*(\pi_1(Y_2, y_2))$. Then

$$p_1_*(\pi_1(Y_1, y_1)) \supset p_2_*(\pi_1(Y_2, y_2)).$$

By Theorem 2.5, $p_1$ has unique lift such that

$$f_1 : (Y_1, y_1) \longrightarrow (Y_2, y_2) \text{ satisfying } p_2 f_1 = p_1.$$
Also
\[ p_2, (\pi_1(Y_2, y_2)) \ni p_1, (\pi_1(Y_1, y_1)) \]
then \( p_2 \) has unique lift such that
\[ f_2 : (Y_2, y_2) \rightarrow (Y_1, y_1) \]
satisfying \( p_1 f_2 = p_2 \).

Then \( p_2 f_1 f_2 = p_2 \) and \( p_1 f_2 f_1 = p_1 \). Since \( f_2 f_1 \) and \( f_1 f_2 \) are uniquely defined, \( f_1 f_2 \) and \( f_2 f_1 \) are both identity maps. So, \( f_1 \) and \( f_2 \) are inverse of each other, so they are the homeomorphism satisfying covering space isomorphism properties. \( \square \)

Now we will state classification theorem of covering spaces and finish its proof.

**Theorem 4.5.** Let \( X \) be path connected, locally path connected and semi-locally simply connected space. Then there exists a bijection between pointed covering spaces of \( X \) and subgroups of \( \pi_1(X, x_0) \). If we ignore base points then there exists a bijection between covering spaces of \( X \) and conjugacy classes of subgroups of \( \pi_1(X, x_0) \).

Now we can prove this theorem mostly with the work we have done before in this project.

**Proof.** First part of theorem is already proved before. To prove the second part we will observe what is the effect of changing base point of a covering space to image of its fundamental group under the induced homomorphism of the projection map.

Let \( p : Y \rightarrow X \) be a covering space of \( X \) where
\[ p_*(\pi_1(Y, y_0)) = H_1 \text{ and } p_*(\pi_1(Y, y_1)) = H_2. \]

Let \([\gamma] \in H_1 \leq \pi_1(X, x_0)\). Then \( \gamma \) has a unique lift which we can denote with \( \tilde{\gamma} \) where \( \tilde{\gamma}(0) = y_0 \). Also, since \( Y \) is path connected there exists a path such that
\[ \tilde{\eta}(0) = y_1 \text{ and } \tilde{\eta}(1) = y_0. \]

Then \([\tilde{\eta}\gamma\tilde{\eta}] \in \pi_1(Y, y_1)\) Therefore, \( p([\tilde{\eta}\gamma\tilde{\eta}]) \in p_*(\pi_1(Y, y_1)) = H_2. \)

Since \( p(y_0) = p(y_1) = x_0, \) \( p\tilde{\eta} = \eta \) and \( p\tilde{\eta} = \tilde{\eta} \)

are both loops in \( X \). Therefore, \( p([\eta\gamma\tilde{\eta}]) = [\eta][\gamma][\tilde{\eta}] \) where \([\eta][\gamma][\tilde{\eta}] \) is conjugate of \([\gamma]\) with the homotopy class of the loop \([\eta] \in \pi_1(X, x_0)\). This shows \( H_2 = [\eta]H_1[\tilde{\eta}] \). So, \( H_1 \) and \( H_2 \) are conjugate to each other. This means if \( Y \) is a path connected space, then for any base point \( y \in Y \), \( p_*(\pi_1(Y, y)) \) will be in this conjugacy class of subgroups and this finishes the proof. \( \square \)

**References**


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