A CATEGORY THEORETICAL APPROACH TO CLASSIFICATION OF COVERING SPACES

ADNAN CİHAN ÇAKAR

Abstract. In this project we continue our Senior Project I topic which was focused on classification of path connected covering spaces. This time we use more category theoretical approach to the subject. We state and prove a classification theorem of covering spaces with no connectedness assumption. This theorem tells that covering space category of a topological space is equivalent to the category of functors from fundamental groupoid of this space to \textbf{Sets}. Also we prove some important corollaries of this theorem which reveals some connections with \(G\)-set theory, in the case the base space is path connected. Moreover, another corollary gives a one-to-one correspondence between path connected covering spaces and subgroups of the fundamental group of a path connected base space. We finish our project giving a criteria for a path connected and locally path connected topological space to have a universal covering.

Introduction

In Senior Project I, we investigated how we can classify covering spaces of a topological space \(X\). In that project, we mainly followed a direct and classical method. We basically constructed a universal cover for \(X\) using path homotopy classes of paths and generated other covers of \(X\) by defining an equivalence relation on this universal cover. Also we used subgroups of the fundamental group of the base space in order to give a classification theorem. However this theory only allows us to classify path connected covering spaces of \(X\).

In this project we continue investigating classification of covering spaces of a topological space \(X\). We mainly use category theory in order to understand covering spaces. After giving some background information about categories and \(G\)-sets we start defining category of covering spaces of \(X\), denoted by \(Cov(X)\). Then we define the fundamental groupoid of \(X\) so that we could construct our classifying category which is \(Func(\pi(X), \textbf{Sets})\) and after that point we will reserve most of the paper to prove \(Func(\pi(X), \textbf{Sets})\) is equivalent to \(Cov(X)\).

For that we need some more constructions. First of them is the monodromy functor of a covering map. Then we define a functor from \(Cov(X)\) to \(Func(\pi(X), \textbf{Sets})\) which we denote by \(\Omega\). This functor \(\Omega\) maps a covering map to its monodromy functor. Definition of this functor on morphism sets will be rather natural.

To prove that \(\Omega\) is an equivalence of categories we first show that every functor in \(Func(\pi(X), \textbf{Sets})\) is the monodromy functor of a covering space. For that we construct a covering map using an arbitrary functor \(F\) in \(Func(\pi(X), \textbf{Sets})\). This construction is the key point of the proof since after that it is easy to see \(\Omega\) gives a bijection between morphism sets and this finishes the proof of the theorem.
After proving this main theorem, we state and prove some corollaries. These corollaries show that if the base space is path connected then our classification category becomes $\text{GSet}$ which means $\text{Cov}(X)$ is equivalent to $\text{GSet}$. Also second corollary shows that there is a one-to-one correspondence between path connected covering spaces of $X$ and subgroups of the fundamental group of $X$, again in the case that $X$ is path connected. This is basically a result follows from $G$-set theory. We finally state and prove a theorem gives an if and only if statement for $X$ to have a universal covering.

We should state that in this project we mainly followed the paper “Fundamental Group and Covering Spaces” by J. M. Møller.

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1. Preliminaries and Definitions

Before we start introducing covering space theory we shall first give some definitions and theorems we refer later in the project. Since we use category theory to construct and prove the main theorem and its corollaries, we first explain some concepts of category theory. We start giving the definition of a category.

**Definition 1.1.** A category is a mathematical object which consists of a set of objects and morphisms between these objects. In a category every morphism has a domain and a codomain. For a category $C$, we denote the set of objects by $\text{obj}(C)$ and for two elements $c$ and $d$ in $\text{obj}(C)$, $\text{Hom}_C(c, d)$ denotes the set of morphisms in $C$ with domain $c$ and codomain $d$. Every object in $C$ has an identity morphism which we denote by $\text{id}_c$ and a composition of two morphism $f$ and $g$ is defined whenever domain of $g$ is equal to codomain of $f$. We denote $f$ composition $g$ with $g \circ f$. Also, in a category $C$ followings are satisfied;

(i) If $f \in \text{Hom}_C(c, d)$ and $g \in \text{Hom}_C(d, e)$ then $g \circ f \in \text{Hom}_C(c, e)$.

(ii) For every $f$, $g$, $h$ morphisms in $C$, with appropriate domains and codomains for compositions, $f \circ (g \circ h) = (f \circ g) \circ h$.

(iii) For every $f \in \text{Hom}_C(c, -)$ and for every morphism $g \in \text{Hom}_C(-, c)$ we have $f \circ \text{id}_c = f$ and $\text{id}_c \circ g = g$.

**Remark 1.2.** A morphism $f \in \text{Hom}_C(c, d)$ is called an isomorphism if there exists a morphism $g \in \text{Hom}_C(d, c)$ such that $f \circ g = \text{id}_d$ and $g \circ f = \text{id}_c$. If there exists an isomorphism between two objects $c$ and $d$ in $C$ then $c$ and $d$ are said to be isomorphic in $C$ and we write $c \cong C d$.

**Example 1.3.** An example of category is $\text{Sets}$ which is the category of all sets with morphisms as functions between these sets. Identity of a set is identity function and composition is usual composition of functions. It is easy to see $\text{Sets}$ satisfy the conditions above with this construction.

This is an important example since we particularly use this category to classify covering spaces. Before we give another important example we give some other definitions which is necessary for this example and we refer in the statement of the main theorem.

**Definition 1.4.** If $C$ and $D$ are two categories, then a functor $F$ consists of two function which we both denote with $F$ for simplicity of notation. First function is
between object sets an the second function is between morphism sets. Also these two functions satisfy following conditions;

(i) For every \( f \in \text{Hom}_C(c, d) \), \( F(f) \in \text{Hom}_C(F(c), F(d)) \).
(ii) \( F(id_c) = id_{F(c)} \) for every object \( c \in C \).
(iii) \( F(f \circ g) = F(f) \circ F(g) \) for every morphisms \( f, g \in C \).

Functors can be seen as morphisms between different categories. However, they can also be seen as objects of another category and in functor category morphisms are called natural transformations. This concept is important as being morphisms of functor category and also we use natural transformations to define equivalence of categories. So, we now give definition of a natural transformation.

**Definition 1.5.** If \( C \) and \( D \) are two categories and \( J, K : C \rightarrow D \) are two functors, then a natural transformation \( \tau : J \Rightarrow K \) is a function which maps every object \( c \in \text{obj}(C) \) to a morphism \( \tau_c \in \text{Hom}_D(J(c), K(c)) \) such that;

\[
\tau_d \circ J(f) = K(f) \circ \tau_c
\]

for every \( c, d \in C \) and for every \( f \in \text{Hom}_C(c, d) \).

Also we should note that if for every object \( c \) in \( C \), \( \tau_c \) is an isomorphism then \( \tau \) is called a natural isomorphism. If such natural isomorphism does exist, \( J \) and \( K \) are said to be naturally isomorphic.

Now we give another important example of category as we mentioned earlier.

**Example 1.6.** If \( C \) and \( D \) are two categories, then \( \text{Func}(C, D) \) is the category whose objects are functors from \( C \) to \( D \) and morphisms are natural transformations between these functors.

Now we constructed some of the basic concepts of category theory which we need in the project and now we give the definition of two categories being equivalent. This definition of equivalence is important since the main theorem of this project claims equivalence of two particular categories and the main object of this project is proving this equivalence and showing some of its consequences.

**Definition 1.7.** Two categories \( C \) and \( D \) are said to be equivalent if there exists functors \( J, K \) as follows;

\[
J : C \rightarrow D \\
K : D \rightarrow C
\]

where \( J \circ K \) and \( K \circ J \) are naturally isomorphic to identity functor respectively.

In this definition we do not require to have \( J \circ K \) and \( K \circ J \) to be equal to identity functor. Only natural isomorphism is enough. The reason is that this definition is quite sufficient for most of the cases and in the category theory, equivalence of categories is quite close to say that two categories are the same. Isomorphism of categories, which is defined similarly with composition of functors are identity functor, is usually considered as too strong.

As we stated before, in this project, we show that two particular categories are equivalent. To show two categories are equivalent we can find such functors satisfying 1.7. However the following theorem provide us an alternative method to show the same thing which we use to prove the main theorem.
Theorem 1.8. If $C$ and $D$ are two categories and $F : C \to D$ is a functor, then $F$ is an equivalence of categories if and only if it satisfies the following two conditions:

(i) For every object $d \in D$ there exists an object $c \in C$ such that $F(c) \cong_D d$.

(ii) $F : \text{Hom}_C(c,d) \to \text{Hom}_D(F(c), F(d))$ is a bijection for every $c, d \in C$.

Here what we mean by $F$ to be an equivalence of categories is that there exists a functor $J$, where $F$ and $J$ satisfy the conditions given in 1.7.

We omit the proof of the theorem which can be found in [3] but we now explain the idea of the proof in one direction since we use this idea later in the project. The idea is to define a functor $J$ which maps an object $d$ of $D$ to any $c$ in $C$ satisfying $F(c) \cong_D d$. Then using the second condition it can be showed $F$ and $G$ satisfy 1.7. Also here we should note that $J$ is unique up to natural isomorphism.

With this theorem we have given all the preliminary definitions and concepts we refer later from the category theory. But before continuing with covering space theory we need to mention another important concept, namely $G$-set, which we also use to classify covering spaces. We show in this project for a path connected base space $X$ covering space category is equivalent to the $G$-set category. Therefore definition of a $G$-set is particularly important and also we use what we know from $G$-set theory to make some conclusion about covering maps. So we start by giving the definition of a $G$-set.

Definition 1.9. Let $G$ be a group. Then a set $X$ is called a right $G$-set if there exists a function

$$
\phi : X \times G \to X
$$

satisfying following conditions;

(i) $x1_G = x$ for every $x \in X$.

(ii) $(xg)h = x(gh)$ for every $x \in X$ and for every $g, h \in G$.

Note that this function $\phi$ is called a right $G$ action on $X$. Since we want to define category of $G$-sets we continue giving the definition of morphisms in this category as follows.

Definition 1.10. If $X$ and $Y$ are two $G$-sets then a morphism between these $G$-sets is a function $f : X \to Y$ satisfying;

$$
f(xg) = f(x)g.
$$

These morphism are called $G$-maps.

Now we can give the definition of the $G$-set category which is the category whose objects are $G$-sets and morphisms are $G$-maps. This category is usually denoted by $G\text{Set}$. $G$-sets are an important object of algebra and as we stated before we use some theorems from this theory to prove some of the classifying arguments. We start with giving definitions of some important $G$-sets.

If a group $G$ acts on a set $X$ from right then for an element $x$ in $X$, $xG$ is called the orbit of $x$ and defined as follows;

$$
xG = \{xg \mid g \in G\}.
$$
Another important definition is the stabilizer of a point. Let $x$ be a point in $X$. Then stabilizer subgroup $\mathcal{G}^x$ of $x$ is defined as follows:

$$\mathcal{G}^x = \{ g \in G \mid xg = x \}.$$  

The reason why stabilizer set is called stabilizer subgroup is that it is in fact a subgroup of $G$. Then we can define the quotient set which is:

$$\mathcal{G}^x \setminus G.$$ 

We should note that this quotient is not necessarily be a group since $\mathcal{G}^x$ need not to be normal in $G$. However both $\mathcal{G}^x \setminus G$ and $\mathcal{G}^x$ are $G$-sets with usual right multiplication. Moreover these two sets are isomorphic in the category $\textbf{GSet}$ [3]. This observation has an important result. First we recall that an $G$-set is called transitive if for every $x, y \in X$ there exists $g \in G$ such that $xg = y$. So, if $X$ is a transitive $G$-set then $\mathcal{G}^x = X$ for every $x \in X$. Then $X$ is isomorphic to the $G$-set, defined as $\mathcal{G}^x \setminus G$ for some $x$ in $X$. So, this relation gives a one-to-one correspondence between subgroups of a group $G$ and transitive $G$-sets. The full-subcategory of $\textbf{GSet}$ generated by all transitive $G$-sets is also referred as the orbit category of $G$ and denoted by $O_G$. This finishes all the statements we would like to give as background information and in the next chapter we start working with covering spaces.

### 2. Covering Spaces

In this section we give the definition of a covering space and we give some other basic definitions and theorems about covering spaces. We state some general theorems about covering spaces which are known in covering space theory. We use these theorems as a background information for our theorems and for our proofs. We start with the definition of a covering space.

**Definition 2.1.** Let $X$ be a topological space. Then a topological space $Y$ with a projection map $p : Y \to X$ is called a covering space if there exists an open cover $\{U_i\}$ of $X$ where for all $U_i$ the preimage of $U_i$ is a disjoint union of open sets in $Y$ and for every such disjoint open set $\tilde{U}_i$,

$$p : \tilde{U}_i \to U_i$$

is a homeomorphism.

This map $p$ is called a covering map. In the rest of the project both covering map and covering space are used to refer to the pair of a covering space and a covering map $(Y,p)$. Also, note that in this definition and for the rest of the project, map refers to a continuous function.

We continue with a simple yet important example. This example is important since it is one of the model examples of the theory.

**Example 2.2.** Let $X = S^1$. Then a covering map of $X$ is defined as follows;

$$p : \mathbb{R} \to S^1$$

$$t \mapsto (\cos(\pi t), \sin(\pi t)).$$
To see this is a covering map let $U_1 = S^1 \setminus (-1, 0)$ and $U_2 = S^1 \setminus (1, 0)$ then,

$$p^{-1}(U_1) = \coprod_{n \in \mathbb{Z}} (n, n + 1) \quad \text{and} \quad p^{-1}(U_2) = \coprod_{n \in \mathbb{Z}} \left(n + \frac{1}{2}, n + \frac{3}{2}\right).$$

Also,

$$p : (n, n + 1) \to S^1 \setminus (1, 0) \quad \text{and} \quad p : \left(n + \frac{1}{2}, n + \frac{3}{2}\right) \to S^1 \setminus (1, 0)$$

are homeomorphisms. [1].

**Remark 2.3.** We also note that

$$p : S^1 \to S^1 \quad \text{defined as} \quad (\cos(\pi t), \sin(\pi t) \mapsto (\cos(k\pi t), \sin(k\pi t))$$

defines a covering space of $S^1$ for any natural number $k$. [1].

This observation brings up the following question.

**Question 2.4.** How can we classify all covering space of a topological space $X$?

To answer this question we define the category of covering spaces of a topological space $X$. Therefore, now we give the definition of morphisms between covering spaces.

**Definition 2.5.** Let $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ be two covering maps over $X$. Then a morphism between these covering maps is a continuous function $f : Y_1 \to Y_2$ satisfying $p_1 = p_2 f$. We call such a function, **covering morphism** for the rest of the project.

Now we are ready to give the definition of the category of covering spaces of $X$.

**Definition 2.6.** The category of covering spaces of $X$ is the category of covering maps over $X$ with covering morphisms between them defined as in 2.5. It is denoted by $Cov(X)$.

At this point we need to state some important properties of covering spaces. These properties are crucial for well-definedness of some functions we define later in the project. Therefore, although these properties are very basic in covering space theory they are rather important for this theory and for our project. But before stating these properties we need to give the definition of a lift.

**Definition 2.7.** Let $p : Y \to X$ be a covering map and $f : Z \to X$ be a map. A lift of $f$ is a map $\tilde{f}$ such that:

$$\tilde{f} : Z \to Y$$

satisfying $f = p\tilde{f}$.

Now we state the unique lifting properties. We omit proofs of these theorems since proofs can be easily found in many of algebraic topology textbooks and proofs of these theorems are not in the focus of this project. Instead we give citations from where these proves can be read.

**Theorem 2.8.** Let $p : Y \to X$ be a covering map over $X$. If $\gamma$ is a path in $X$ and $y$ is a point in $p^{-1}(x)$ then there exist a unique path $\tilde{\gamma}$ such that $\tilde{\gamma}$ is a lift of $\gamma$ and $\tilde{\gamma}(0) = y$. 
Proof. [1]. □

**Theorem 2.9.** Let \( p : Y \to X \) be a covering map over \( X \). If \( h \) is a path homotopy in \( X \) and \( \tilde{h}_0 \) is a lift of \( h(t,0) \), then there exist a unique path homotopy \( \tilde{h} \) such that \( \tilde{h} \) is a lift of \( h \) and \( \tilde{h}|_0 = \tilde{h}_0 \).

Proof. [1]. □

With these definitions and theorems we finish stating basic concepts of covering space theory which we use for the rest of the project. At this point we leave covering spaces theory to give definitions which we also use to classify covering spaces.

But before giving the definition of the fundamental groupoid we now explain what we mean by a **path homotopy** and a **path homotopy class of paths**. A path homotopy is a homotopy between two paths which fixes endpoints of paths and a path homotopy class of paths is an equivalence class of a path with respect to path homotopy equivalence.

Now we give the definition of the fundamental groupoid of a topological space \( X \).

**Definition 2.10.** Let \( X \) be a topological space. Then the **fundamental groupoid** of \( X \) denoted by \( \pi (X) \) is the category whose objects are points in \( X \) and morphisms are path homotopy classes of paths between points. In this category identity morphism of \( x \) in \( X \) is the path homotopy class of the constant path at \( x \) and composition of two path homotopy classes of paths is defined as follows;

\[
[\eta] \circ [\gamma] = [\gamma \cdot \eta]
\]

where

\[
\gamma \cdot \eta = \begin{cases} 
\gamma(2t) & 0 \leq t \leq \frac{1}{2} \\
\eta(2t - 1) & \frac{1}{2} \leq t \leq 1 
\end{cases}
\]

**Remark 2.11.** \( \pi (X) \) is a groupoid with this definition.[2]. Which means every morphism is an isomorphism. In \( \pi (X) \) inverse morphism of a path homotopy \( [u] \) is defined as \( [\bar{u}] \) where

\[
\bar{u} = u(1 - t) \text{ for all } t \in [0,1].
\]

In our main theorem we set some restrictions to the base space. We will use these properties of the base space to prove our theorems. Also in proofs, we see that these properties are crucial for our theory. For that reason we now introduce the following topological properties of a topological space.

**Definition 2.12.** A topological space \( X \) is said to be **locally path connected** if for all \( x \in X \) and an open neighbourhood \( U \) of \( x \), there exist a path connected open subset of \( U \) which contains \( x \).

**Definition 2.13.** A topological space \( X \) is called **semi-locally simply connected** if for all \( x \in X \) and for all open neighbourhoods \( U \) of \( x \), there exists an open subset \( V \) of \( U \) such that induced map of natural inclusion

\[
i_* : \pi_1(V,x) \hookrightarrow \pi_1(X,x)
\]

is a trivial map.
Here $\pi_1(X, x)$ denotes the fundamental group of $X$ at point $x$, which is the group of path homotopy classes of loops at $x$. Also it can be defined as the full subcategory of $\pi(X)$ generated by one point $x$. Since $\pi(X)$ is a groupoid, $\pi_1(X, x)$ is a group and this also explains the name and the notation.

In the next section we state and prove the main theorem.

3. The Main Theorem

The main theorem of this project is the following and in this section we prove this theorem.

**Theorem 3.1.** Let $X$ be a locally path connected, semi-locally simply connected topological space. Then $\text{Cov}(X)$ is equivalent to the functor category $\text{Func}(\pi(X), \text{Sets})$.

Before we give proof of this theorem now we briefly explain the idea of the proof. In this proof we use Theorem 1.8. Recall that this theorem tells that to prove two categories are equivalent we need to find a functor which satisfies two conditions stated in the theorem. So, we first define this functor, then we show this functor we define satisfies these two conditions. Therefore this proof can be considered as three main parts followed by each other.

We start giving the definition of the functor from $\text{Cov}(X)$ to $\text{Func}(\pi(X), \text{Sets})$. But to define this functor we need to give some other definitions first.

**Definition 3.2.** If $p : Y \rightarrow X$ is a covering map, then monodromy functor of $p$ is a functor from fundamental groupoid of $X$ to $\text{Sets}$ which maps a point $x \in X$ to the set $p^{-1}(x)$ and a path homotopy class of a path $[u]$ from $x_1$ to $x_2$ to the function $f_{[u]}$ defined as follows;

$$f_{[u]} : p^{-1}(x_1) \rightarrow p^{-1}(x_2)
\quad y \mapsto \hat{u}_y(1)$$

where $\hat{u}_y$ denotes for lift of $u$ starting at $y$. Note that the monodromy functor of $p$ is denoted by $F(p)$.

We define the functor from $\text{Cov}(X)$ to $\text{Func}(\pi(X), \text{Sets})$ on object set as the function which maps a covering space to its monodromy functor. But first we need to prove the above definition of a monodromy functor in fact defines a functor. Next theorem shows that.

**Theorem 3.3.** If $p : Y \rightarrow X$ is a covering map, then monodromy functor defined as in Definition 3.2 is a functor.

**Proof.** Let $[u]$ be the path homotopy class of a path $u$ between $x_1$ and $x_2$. Then $F(p)[u]$ maps every $y \in p^{-1}(x)$ to a point in $p^{-1}(x_2)$ since lift $\hat{u}$ of $u$ starting at $y$ exists and $\hat{u}(1) \in p^{-1}(x_2)$.

Also, for a point $x \in X$, $id_x$ is the constant path at $x$. Lift of the constant path at a point $x$ to a point $y \in p^{-1}(x)$ is the constant path at $y$ so $F(id_x)$ maps every $y \in p^{-1}(x)$ to itself. So, $F(id_x)$ is identity function of $p^{-1}(x)$ which is $id_{F(x)}$ by definition.

Now let $[u]$ be a path homotopy class of paths between $x_1$ and $x_2$. Also let $[v]$ be a path homotopy between $x_2$ and $x_3$. For a point $y \in p^{-1}(x_1)$ let $\hat{u}$ be the lift of $u$ and
\( \tilde{v} \) be the lift of \( v \) such that
\[
\tilde{u}(0) = y \text{ and } \tilde{v}(0) = \tilde{u}(0).
\]
By definition \( F(p)[u](y) = \tilde{u}(1) \) and \( F(p)[v](\tilde{u}(1)) = \tilde{v}(1) \). Also clearly \( \tilde{u}\tilde{v} \) is a lift of \( uv \) starting at \( y \) so, \( F([uv])(y) = \tilde{v} \) since \( uv \) has a unique lift starting at \( y \).

Therefore monodromy functor is a functor. \( \square \)

Now we explain how the functor from \( Cov(X) \) to \( Func(\pi(X), \text{Sets}) \) is defined on morphism sets. Let \( f \in \text{Hom}_{Cov(X)}(p_1, p_2) \) then this functor maps \( f \) to a natural transformation \( \tau : F(p_1) \Rightarrow F(p_2) \) defined as \( \tau_x = f \big|_{p_1^{-1}(x)} \). Again we first need to show that this definition makes sense which means \( \tau \) is a natural transformation.

**Theorem 3.4.** If \( p_1 : Y_1 \rightarrow X \) and \( p_2 : Y_2 \rightarrow X \) are covering maps and \( f : Y_1 \rightarrow Y_2 \) is a covering morphism, then \( \tau : F(p_1) \Rightarrow F(p_2) \) defined as \( \tau_x = f \big|_{p_1^{-1}(x)} \) is a natural transformation.

**Proof.** Let \( p : Y \rightarrow X \) be a covering map. Also let \( u \) is a path in \( X \) and \( y \in p^{-1}(x) \). Recall that \( \hat{u}_y \) denotes the lift of \( u \) to \( Y \) starting at \( y \). Since in this theorem there exists two different covering maps we should be careful about notation and also consider this lift as the lift to whichever covering space \( y \) is a point of.

Now let \( x_1, x_2 \) in \( X \) and \([u]\) is a path homotopy between these points. Then monodromy functor \( F(p_1) \) and \( F(p_2) \) maps points and this morphism as follows;
\[
F(p_1)(x_1) = p_1^{-1}(x_1) \quad F(p_1)(x_2) = p_1^{-1}(x_2) \\
F(p_2)(x_1) = p_2^{-1}(x_1) \quad F(p_2)(x_2) = p_2^{-1}(x_2).
\]
Also;
\[
F(p_1)([u]) : p_1^{-1}(x_1) \rightarrow p_1^{-1}(x_2) \\
y \mapsto \hat{u}_y(1)
\]
and
\[
F(p_2)([u]) : p_2^{-1}(x_1) \rightarrow p_2^{-1}(x_2) \\
y \mapsto \hat{u}_y(1).
\]
Now let \( f \) be a covering morphism and \( \tau_x = f \big|_{p_1^{-1}(x)} \) and let \( y \in Y_1 \). By definition,
\[
\left( \tau_x, F(p_1)[u] \right)(y) = f\hat{u}_y(1).
\]
Then, \( p_2(f\hat{u}_y) \) is a path in \( X \). Also since \( f \) is covering morphism \( p_2f = p_1 \). Then this is equal to \( p_1\hat{u}_y \). Now we know;
\[
u = p_1\hat{u}_y = p_2f\hat{u}_y.
\]
If we lift both side of this equation to \( f(y) \) then we get, \( f(\hat{u}_y)(1) = \hat{u}_{f(y)}(1) \) by the uniqueness of the lift. Then by definition, \( \hat{u}_{f(y)}(1) = F(p_2)\tau_{x_2} \) and this finishes the proof. \( \square \)

At this point, we completed the definition of the functor from \( Cov(X) \) to the functor category \( Func(\pi(X), \text{Sets}) \). We denote this functor by \( \Omega \) for the rest of the project. We need to show that this definition defines a functor but it is easy to see this fact, since this functor maps morphism to their restrictions. So property of respecting to the
composition and the identity is easily follows from this definition and it only remains to show that this functor is well-defined on morphism sets. But this is also showed by Theorem 3.4.

In the rest of this chapter we prove that this functor satisfies the two conditions given in 1.8. For that we first, show that every functor in $\text{Func}(\pi(X), \text{Sets})$ is monodromy functor of a covering map and then we will show that this functor is bijective on every Hom-Set. In order to show the first condition is satisfied we start with an arbitrary functor $F$ from $\text{Func}(\pi(X), \text{Sets})$ and using $F$, construct a covering map whose monodromy functor is $F$. Now we start this construction. Let $F$ be a functor in the category $\text{Func}(\pi(X), \text{Sets})$. Then we define,

$$Y(F) = \coprod_{x \in X} F(x)$$

and the function,

$$p : Y(F) \to X$$

$$y \mapsto x \quad \text{for all } y \in F(x).$$

Note that $p : Y(F) \to X$ is the covering map we want to construct. However, we still need to give the definition of topology on this set. What we do is to define a topology on this set $Y(F)$ which will make $p$ a covering map. We should remind that the topological space $X$ we are working is semi-locally simply connected and locally path connected. We will use these properties to define this topology. So, this construction shows why these properties are essential for our theory. In order to define this topology, we first need the following proposition.

**Proposition 3.5.** Let $x$ be a point in $X$ and $U$ is an open neighbourhood of $x$ satisfying the following two conditions which we will call principal conditions for the rest of the project;

- (i) $U$ is path connected.
- (ii) The induced map of the inclusion map $i_*(\pi_1(U, x)) \hookrightarrow \pi_1(X, x)$ is trivial.

Then the following function $\rho$ is a bijection.

$$\rho : U \times F(x) \to p^{-1}(U)$$

$$(z, y) \mapsto F([u_z])y$$

where $u_z$ is a path from $x$ to $z$ in $U$ and $p$ is defined as before using the functor $F$.

**Proof.** If $y \in p^{-1}(U)$ then $p(y)$ is in $U$, so there exists a point $z$ in $U$ such that $y$ is an element of $F(z)$. Since $U$ is path connected there exists a path $u$ from $x$ to $z$. Then,

$$\rho(z, F([\bar{u}](y))) = (F([u]) \circ F([\bar{u}]))(y) = y.$$ 

So, $\rho$ is surjective.

If $\rho(z_1, y_1) = \rho(z_2, y_2)$ then $F([u_{z_1}]) (y_1) = F([u_{z_2}]) (y_2)$ so $z_1 = z_2$ since $F([u_{z_1}]) (y_1)$ is in $F(z_1)$ and $F([u_{z_2}]) (y_2)$ is in $F(z_2)$ by definition of codomain of $F([u_{z_1}])$ and $F([u_{z_2}])$. Also, $z_1 = z_2$ implies $[u_{z_1}] = [u_{z_2}]$ since $U$ is null-homotopic. Since $[u_{z_1}]$ is a isomorphism $F([u_{z_1}])$ is a bijection, so $y_1 = y_2$. Therefore $\rho$ is injective. \qed
We will denote the set $\rho(U \times \{y\})$ by $(U, y)$ where $U$ is an open subset of $X$ satisfying principal conditions and $y$ is in $F(x)$ for some $x$ in $U$. Also, we claim all subsets of $Y(F)$ in the form of $(U, y)$ defines a base for topology of $Y(F)$. We will prove this using the fact that all subsets of $X$, satisfying the principal conditions, defines a topological basis for the topology of $X$ since $X$ is locally path connected and semi-locally simply connected. After proving our claim we set the topology generated by this family of subsets as the topology of $Y(F)$. Before we state our claim as a proposition, we now prove a lemma which we use to prove our claim.

**Lemma 3.6.** Let $(U, y)$ and $(U, z)$ be subsets of $Y(F)$ as defined before. If intersection of $(U, y)$ and $(U, z)$ is not empty then $(U, y) = (U, z)$. Also if $W \subset U$ then $(W, w) \subset (U, w)$

**Proof.** If $w \in (U, y) \cap (U, z)$ then by the definition of $(U, y)$ and $(U, z)$,

$$w = F([u_{x_1}]) (y) = F([u_{x_2}]) (z)$$

for some $x_1$ and $x_2$ in $U$. Then $F([u_{x_1}, u_{x_2}]) (y) = z$. So, $z \in (U, y)$ which implies $(U, z) \subset (U, y)$. The other inclusion can be proved similarly.

If $z$ is an element of $(W, w)$ then $F([u_z]) (w) = z$ for some $x$ in $W$. Since $w$ and $u_z$ is also in $U$, $z \in (U, w)$ by definition. So, $(W, w) \subset (U, w)$. \hfill \Box

**Proposition 3.7.** If $X$ is a topological space and $F$ is a functor from $\pi(X)$ to $\text{Sets}$. Then all subsets $(U, y)$ of $Y(F)$ defined as before is a basis for topology of $Y(F)$.

**Proof.** We need to show this family of subsets satisfies two conditions to be a topological basis. First, if $z$ is in $Y(F)$ then $z$ is an element of $F(x)$ for some $x$ in $X$. So, $x$ is in a subset $U$ of $X$ satisfying the principle conditions. Hence, $z \in (U, z)$ since $\rho(x, z) = z$.

Now let $w \in (U, y) \cap (V, z)$. Then $p(w) \in U \cap V$ and there exists a $W$ subset of $X$ satisfying principle conditions. So, $(W, w)$ is also a basis element of $Y(F)$ with previous definition. Then $w \in (W, w)$. Also,

$$(W, w) \subset (U, w) = (U, y) \text{ and } (W, w) \subset (V, w) = (V, z).$$

So, $w \in (W, w) \subset (U, y) \cap (V, z)$. Therefore the set of all such $(U, y)$ is a topological basis. \hfill \Box

This completes the proof of our claim and the next theorem shows why we introduce especially this topology on $Y(F)$.

**Theorem 3.8.** Let $p : Y(F) \to X$ is the map defined as before. If we define the topological space $Y(F)$ with the topology generated by all $(U, y)$ where $U$ satisfies principle conditions and $y$ is in $F(x)$ for some $x$ in $U$, then $p$ is covering map.

**Proof.** We set open cover of $X$ as all $U$ open subsets of $X$ satisfying the principal conditions. This is in fact an open cover since such subsets defines a basis for $X$. Now recall that $\rho$ is the function defined as follows in 3.5;

$$\rho : U \times F(x) \to p^{-1}(U)$$

$$(z, y) \mapsto F([u_z])(y)$$

and it is a is bijection. Then,

$$p^{-1}(U) = \rho(U \times F(x))$$
for some(any) $x$ in $U$. Also we know this is equal to
\[
\prod_{y \in F(x)} (U, y).
\]

Therefore $p^{-1}(U)$ is a disjoint union of open sets in $Y(F)$ with this topology. So, the last thing we need to show is $p : (U, y) \to U$ is an homeomorphism for a fixed $y$. To show that we define the following map;
\[
\rho |_y : U \to (U, y) \\
x \mapsto \rho(x, y).
\]

Then this is a bijection since $\rho$ is a bijection. Also we know $p \circ \rho |_y = \text{id}_U$, so $p^{-1} = \rho |_y$ for $p$ defined only on $(U, y)$ for a fixed $y$. To show $\rho |_y$ is a homeomorphism we shall show this mapping is open and continuous. Let $V$ open subset of $U$. Then $\rho |_y (V) = (V, y)$ is an open subset of $(U, y)$. Also if $(V, y)$ is an open subset of $(U, y)$ then preimage of $(V, y)$ is $V$ which is open and subset of $U$. Therefore for a fixed $y, (U, y) \cong U$ with the map $p$ and also this shows $p$ is continuous, since for any open subset $(U, y)$ of $Y(F)$ and a point $x$ in the preimage of $(U, y)$ we can find an open neighbourhood of $x$ using this homeomorphism which is open and still in the preimage. Therefore, $p$ is a covering map. \qed

This shows that starting with an arbitrary functor in $\text{Func}(\pi(X), \text{Sets})$ we can construct a covering map. So, in fact we can define a functor from $\text{Func}(\pi(X), \text{Sets})$ to the $\text{Cov}(X)$ on object set as the function which maps a functor $F$ to $Y(F)$. But we do not use this functor particularly to prove our main theorem. However, we want to state it is possible after defining this second functor also on Hom-Sets and then show that these two functors are category equivalence, but as we said before we will use 1.8 to prove the main theorem. To use this theorem we will use the functor $\Omega$. So, next theorem shows this functor satisfies the first condition of 1.8.

**Theorem 3.9.** If $F$ is a functor in $\text{Func}(\pi(X), \text{Sets})$, then $F$ is the monodromy functor of $Y(F)$.

**Proof.** Let $p : Y(F) \to X$ be the covering map as defined before. Then monodromy functor of $p$ denoted by $F(p)$ maps a point $x$ in $X$ to $p^{-1}(x)$ which is equal to $F(x)$ since $p(y) = x$ if and only if $y \in F(x)$ by definitions of $p$ and the space $Y(F)$.

Also we need to show that $F([u]) = F(p)([u])$. To show that we first assume $u$ is in an open subset $U$ of $X$ satisfying the principle conditions. Then let $[u]$ is the path homotopy class of the path $u_\gamma$ in $X$ from $u(0)$ to $u(t)$ and travels along the path $u$. Note that this path also can be described as $u_t(s) = u(st)$. Then for $u$ we define the following path in $Y(F)$;
\[
\gamma : I \to Y(F) \\
t \mapsto F([u_t])y.
\]

First we see that $p(\gamma) = u$ since $F([u_t])(y)$ is in $F([u_t])$ so $p(F([u_t])y) = u(t)$. Also this path is continuous since $F([u_t])(y) = \rho |_y (u(t))$, where $u$ is a path and $\rho$ is a homeomorphism if we fix $y$ as we showed in the proof of the Theorem 3.8. Therefore,
\( \gamma \) is a lift of \( u \) and \( F([u]) = y \). So,
\[
F(p)y = F([u_1])y = F([u])y.
\]

Now we will extend this result to any \([u]\) path homotopy class (not necessarily in one \( U \) satisfying the principle conditions). First let \( x \in X \) then we define \( U_x \) as the open neighbourhood of \( x \) satisfying principle conditions. Then we define \( I_x = u^{-1}(U_x) \). \( \{I_x\}_{x \in X} \) will cover of \( I \) and since \( I \) is compact there exists a finite subcover of \( I_x \). Then let \( I_i \) be this subcover where \( i = 1, 2, \ldots n \). Then we can find \( u_i \)'s such that \( u = u_1u_2 \ldots u_n \) and \( u_i \) is contained in \( U_i \) completely. Also we note that \( F([u_1]) \circ F([u_2]) = F([u_1u_2]) \) since \( F \) is a functor. So,
\[
F([u]) = F([u_1u_2 \ldots u_n]) = F([u_n]) \circ \ldots \circ F([u_1]).
\]

Since every \( u_i \) is in \( U_i \) satisfying principle condition we can use the previous result so,
\[
F([u])y = F([u_1u_2 \ldots u_n])(y) = F([u_n]) \circ \ldots \circ F([u_1])(y)
\]
and this is equal to
\[
\left( F(p)[u_n] \circ \ldots \circ F(p)[u_1] \right) = (F(p)[u])(y).
\]

So, this finishes the proof.

This theorem shows that every functor in the category \( Func(\pi(X), \text{Sets}) \) is the monodromy functor of a covering space. Therefore every object in the functor category is isomorphic to image of a covering space under the functor \( \Omega \).

As the last part of the proof we show that this functor gives a bijection between morphism sets. To show that we define the inverse of this functor on Hom-Sets and we prove this function is well defined which means its image is a covering morphism.

**Lemma 3.10.** Let \( F(p_1) \) and \( F(p_2) \) be monodromy functors of two covering maps \( p_1 : Y_1 \rightarrow X \) and \( p_2 : Y_2 \rightarrow X \). Also let \( \tau \) a natural transformation between these two functors. Then the function \( f \) defined as follows;
\[
f : Y_1 \rightarrow Y_2
\]
\[
y \mapsto \tau_x(y) \text{ for all } y \in F(x)
\]
is a covering morphism.

**Proof.** Since \( \tau \) is a natural transformation we know for every \( x \in X \), \( \tau_x \) is a function from \( F(p_1)(x) \) to \( F(p_2)(x) \). So, if \( p_1(y) = x \) then \( y \in F(p_1)(x) \). So,
\[
f(y) = \tau_x(y) \in F(p_2)(x).
\]

Therefore, \( p_2f(y) = x = p_1(y) \).

Next we show that this function is continuous. For that let \( V \) be an open set in \( Y_2 \). Also let \( y \in f^{-1}(V) \). This implies \( f(y) \in V \). Let \( p_2f(y) = p_1(y) = x \). Then there exists an open subset \( U \) of \( X \) containing \( x \) such that \( p_1^{-1}(U) \) and \( p_2^{-1}(U) \) is a disjoint union of open sets in \( Y_1 \) and \( Y_2 \) respectively, where each of them are homeomorphic to \( U \) and \( U \) satisfies principal conditions. We know that in the preimage of \( U \) with respect to \( p_1 \), one open component of disjoint union contains \( y \). Let \( \widetilde{U}_1 \) be the open subset of \( Y_1 \) which is a disjoint component of the preimage of \( U \) containing \( y \). Similarly let \( \widetilde{U}_2 \) is the open subset of \( Y_2 \) which is a disjoint component of the preimage of \( U \) with respect to
\[ p_2 \text{ which contains } f(y). \] Without loss of generality we assume \( \widetilde{U}_2 \) is completely inside \( V \) since otherwise we can use \( p_2(V \cap \widetilde{U}_2) \) as the neighbourhood of \( x \). Now our claim is \( \widetilde{U}_1 \) is in \( f^{-1}(V) \).

Let \( z \in \widetilde{U}_1 \). Since \( \widetilde{U}_1 \) is homeomorphic to \( U \) it is path connected. Let \( \widetilde{u}_1 \) be a path between \( y \) and \( z \). Then \( p_1\widetilde{u}_1 = u \) is a path in \( U \). Also let

\[ p_1(y) = x_1 \text{ and } p_1(z) = x_2. \]

Then \( f(z) = \tau_{x_2}(z) \). But since \( \tau \) is a a natural transformation

\[ \tau_{x_2} \circ F(p_1)[u] = F(p_2)[u] \circ \tau_{x_1}. \]

Also \( F(p_1)[u](y) = z \) by the definition of monodromy functor. Then

\[ f(z) = F(p_2)[u] \circ \tau_{x_1}(y). \]

This is equal to the end point of the lift of \( u \) to \( Y_2 \) starting at \( f(y) \) since

\[ \tau_{x_1}(y) = f(y). \]

We know there exists \( z' \in \widetilde{U}_2 \) such that \( z \in p_2^{-1}(x_2) \) and since \( \widetilde{U}_2 \) is also path connected there exist a path \( \widetilde{u}_2 \) from \( f(y) \) to this point \( z' \). Moreover if \( p_2\widetilde{u}_2 \) is not homotopic to \( u \) this gives a contradiction with the fact that \( U \) is null homotopic. So, lift of \( u \) starting at \( f(y) \) is \( \widetilde{u}_2 \) and it has ending point in \( \widetilde{U}_2 \). So, \( f(z) \in \widetilde{U}_2 \) which shows

\[ \widetilde{U}_1 \subset f^{-1}(\widetilde{U}_2) \subset f^{-1}(V). \]

So, \( f^{-1}(V) \) is open and therefore \( f \) is continuous. \( \square \)

At this point we have two functions on Hom-Sets of categories. We will show these are inverse of each other, so they are bijections and this will be end of the proof of the main theorem. Next theorem shows these two functions are inverse functions.

**Theorem 3.11.** Let \( p_1 : Y_1 \to X \) and \( p_2 : Y_2 \to X \) are two covering map. If \( \psi \) and \( \varphi \) are the functions defined as follows;

\[
\psi : Hom_{\text{Cov}(X)}(p_1, p_2) \to Hom_{\text{Func}(\pi(X), \text{Sets})}(F(p_1), F(p_2))
\]

\[
f : p_1 \to p_2 \mapsto \tau : F(p_1) \Rightarrow F(p_2)
\]

\[
\tau_x = f \big|_{p^{-1}(x)}
\]

and

\[
\varphi : Hom_{\text{Func}(\pi(X), \text{Sets})}(F(p_1), F(p_2)) \to Hom_{\text{Cov}(X)}(p_1, p_2)
\]

\[
\tau : F(p_1) \Rightarrow F(p_2) \mapsto f : p_1 \to p_2
\]

\[
y \mapsto \tau_y(y) \text{ where } y \in F(x)
\]

then \( \psi \) and \( \varphi \) are inverse functions.

**Proof.** We can directly calculate composition of these two functions. First let \( f \) in \( Hom_{\text{Cov}(X)}(p_1, p_2) \). Then,

\[
\left( (\varphi \circ \psi) f \right)(y) = (\psi f)_{p_1(y)}(y) = f \big|_{p_1^{-1}(p_1(y))} (y) = f(y).
\]
Also,
\[
\left( (\psi \circ \varphi) \tau \right)_x = \varphi \tau \mid_{p^{-1}(x)} = \tau_x
\]
from definition of \( \varphi \). So, this shows these two functions are inverse to each other which means they are bijections.

So, this finishes the proof of the main theorem. As we stated before this theorem has some important corollaries. We state and prove those corollaries in the next section.

4. Corollaries of The Main Theorem

Now we state some corollaries of this theorem which are also indicators of the importance of this theorem. Also this corollaries explains how this theorem classify covering spaces of a topological space using concepts we are rather familiar from algebra.

**Corollary 4.1.** If \( X \) is locally path connected, semi-locally simply connected and path connected topological space, then \( \text{Cov}(X) \) is equivalent to \( \text{GSet} \). \( (G = \pi_1(X, x_0) \) in this notation)\]

**Proof.** Let \( X \) be a path connected, locally-path connected and semi-locally simply connected topological space. Then we define the inclusion functor from \( \pi_1(X, x_0) \) to the category \( \pi(X) \).

\[ i : \pi_1(X, x_0) \to \pi(X) \]
\[ x_0 \mapsto x_0 \]
\[ [u] \mapsto [u] \text{ where } u \text{ is a loop starting at } x_0. \]

Then this functor \( i \) is a category equivalence since every \( x \) in \( X \) is isomorphic to \( x_0 \) and \( i \) gives a bijection between morphism sets. Therefore \( \text{Func}(\pi_1(X, x_0), \text{Sets}) \) is equivalent to \( \text{Func}(\pi(X), \text{Sets}) \). Also since \( \pi_1(X, x_0) \) is a group, \( \text{Func}(\pi_1(X, x_0), \text{Sets}) \) is the category of \( G \)-objects. Then, since its objects are functors to the \( \text{Sets} \), it is the category of \( G \)-sets where \( G \) denotes \( \pi_1(X, x_0) \).

This corollary shows that if \( X \) is path connected, then \( \text{GSet} \) is equivalent to the \( \text{Cov}(X) \). This gives another important classification of the covering spaces. This restriction in this corollary does not lessen the importance of this corollary because for a disconnected space \( X \), we can understand covering spaces of \( X \) by understanding covering spaces of path components. Also it gives a connection between covering space theory and \( G \)-set theory which we have some understanding coming from algebra.

The next corollary is also significant since it gives a one-to-one correspondence between path connected covering spaces of a path connected base space and subgroups of the fundamental group of the base space. So, knowing all subgroup of the fundamental group of the base space we know all path connected covering spaces of this path connected base space. So, we continue stating this second corollary of the main theorem.

**Corollary 4.2.** If \( X \) is locally path connected, semi-locally simply connected and path connected, then there is a one-to-one correspondence between path connected covering spaces over \( X \) and subgroups of \( \pi_1(X, x_0) \).
To prove the second corollary we make the following observation. From 4.1 we know there is a one-to-one correspondence between $G$-sets and covering maps. This connection follows from the proof of the main theorem and corollary one.

Recall that for a path connected space the functor $\Omega$ which maps a covering space to its monodromy functor is a category equivalence. Also from the proof of 4.1 natural inclusion from $\pi$ to its monodromy functor is a category equivalence. This means, we can define a category equivalence from $\text{Func}(\pi(X), \text{Sets})$ to $\text{Func}(\pi_1(X,x), \text{Sets})$ which maps a functor $F$ in $\text{Func}(\pi(X), \text{Sets})$ to $F \circ i$ where $i$ is natural inclusion functor. Therefore, the composition of these functors is category equivalence and which is the functor maps a covering space to the $\pi_1(X,x)$-set defined as follows;

$$\phi : p^{-1}(x) \times G \to p^{-1}(x)$$

$$(y, [u]) \mapsto F(p([u]))(y)$$

where $[u] \in \pi_1(X,x)$. Moreover, next lemma gives a connection between path connectedness of the covering space and transitivity of this action.

**Lemma 4.3.** Let $X$ be a path connected, locally path connected and semi-locally simply connected topological space and $p : Y \to X$ a covering map over $X$. Then $Y$ is path connected if and only if the action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ defined using monodromy functor as follows;

$$y[u] = F([u])(y) \text{ where } y \in p^{-1}(x) \text{ and } [u] \in \pi_1(X, x_0)$$

is transitive.

**Proof.** If $Y$ is path connected then for every $y_1$ and $y_2$ there exists a path $u$ from $y_1$ to $y_2$. Then $F([p(u)])(y_1) = y_2$ follows from the unique lifting property.

Conversely, if this action is transitive then let $y_1, y_2 \in Y$. Since $X$ is path connected there exists a path $u$ from $p(y_1)$ to $p(y_2)$ and $(F(p)[u])(y_1) \in p^{-1}(p(y_2))$. Let $(F(p)[u])(y_1) = y_3$ then since action of the fundamental group of $X$ on $p^{-1}(p(y_2))$ is transitive, there exists a path homotopy class $[v] \in \pi_1(X,p(y_2))$ such that

$$(F(p)[v])(y_2) = y_3.$$ 

Therefore $(F(p)[uv])y_1 = y_2$ which means a lift of $uv$ is a path from $y_1$ to $y_2$. \[\square\]

From this lemma and previous observation we know there is a one-to-one correspondence between transitive $\pi_1(X,x)$-sets and path connected covering spaces of $X$. Then we get a one-to-one correspondence between subgroups of $\pi_1(X,x)$ and path connected coverings since we know from $G$-set theory there is a one-to-one correspondence between transitive $G$-sets and subgroups of $G$.

The universal cover of $X$ is a simply connected covering space of $X$. As a last remark we state an if and only if condition for a path connected and locally path connected topological space to have a universal cover. This covering is important for the theory since every path connected covering space can be deduced by adding an equivalence relation on this universal cover. Next theorem states this criteria for a path connected and locally path connected space $X$ to have a universal cover.

**Theorem 4.4.** A path connected and locally path connected topological space $X$ has a universal cover if and only if $X$ is semi-locally simply connected.
Before proving this theorem we make an observation using the next theorem. This observation gives us an important relation between path homotopy classes of paths of the base space and path homotopy classes of paths of a covering space.

**Theorem 4.5.** Let \( p : Y \to X \) be a covering map. Then
\[
\pi(Y) = \pi(X) \rtimes F(p)
\]
where \( \pi(X) \rtimes F(p) \) denotes the Groethendieck construction of \( \pi(X) \) with monodromy functor \( F(p) \) of the covering map.

Before we prove this theorem we will give the definition of the Groethendieck construction and we will see that proof of this theorem is almost direct consequence of this definition.

**Definition 4.6.** Let \( C \) be a category and \( P \) is a functor from \( C \) to \( \text{Cat} \) which is the category of all categories. Then Groethendieck construction which is denoted by \( C \rtimes P \) is the category defined as follows;

(i) Objects of \( C \rtimes P \) are pairs \( (c,y) \) where \( y \) is an object in \( P(c) \).

(ii) Morphism of \( C \rtimes P \) between two objects \( (c,y) \) and \( (d,z) \) are pairs \( (f,w) \) where \( f \) is a morphism between \( c, d \) and \( w \) is a morphism between \( y \) and \( P(f)(z) \).

Now we will prove the theorem.

**Proof of 4.5.** We consider a set as a discrete category which means only morphisms are identities. Then we define functors \( \theta \) and \( \vartheta \) as follows;
\[
\theta : \pi(Y) \to \pi(X) \rtimes F(p) \\
y \mapsto (p(y), y) \\
[u] \mapsto (p(u), \text{id}_{u(1)}) \text{ and }
\vartheta : \pi(X) \rtimes F(p) \to \pi(Y) \\
(x, y) \mapsto y \\
([v], \text{id}_{y}) \mapsto \bar{w} \text{ where } w \text{ is lift of } \bar{u} \text{ to the point } y.
\]
These two maps are bijections because of the unique path lifting. \( \square \)

Also since we consider sets as a category with only identity morphism, definition of Groethendieck construction gives us the following relation;
\[
\text{Hom}_{\pi(Y)}(y_1, y_2) = \{ u \in \pi(X) \mid \text{ such that } (F(p)[u])y_1 = y_2 \}.
\]

Now with this observation we will prove the theorem 4.4 .

**Proof of 4.4.** First assume \( X \) has a universal covering \( \tilde{X} \). If \( x \in X \), then there exist an open neighbourhood of \( x \) such that, preimage of \( U \) is disjoint union of open sets in \( \tilde{X} \) and \( U \) is homeomorphic to each disjoint open set by \( p \). So let \( \tilde{U} \) be one of these sets. Also let \( [u] \in \pi_1(X, x) \). Then, since \( \tilde{X} \) is simply connected, \( p^{-1}[u] \) is homotopic to the constant map at \( y \in p^{-1}(x) \) in \( \tilde{X} \). Then its image is also homotopic to the constant map at \( x \). Also since \( p : \tilde{U} \to U \) is a homeomorphism, \( i = p \circ p^{-1} \). So, \( i_\ast([u]) \) is constant map at \( x \) which implies that \( X \) is semi-locally simply connected.
Conversely, let’s assume $X$ is semi-locally simply connected. Then since $X$ is also locally path connected for every functor $F$ in $\text{Func}(\pi(X), \text{Sets})$ there is a covering space whose monodromy functor is $F$. Then let functor $F$ is defined as follows for a fixed point $x_0 \in X$;

$$F : \pi(X) \to \text{Sets}$$

$$x \mapsto \text{Hom}_{\pi(X)}(x_0, x)$$

$$[u] : \mapsto f_u : \text{Hom}_{\pi(X)}(x_0, x_1) \to \text{Hom}_{\pi(X)}(x_0, x_2)$$

$$[v] \mapsto [uv]$$

Then monodromy functor of $Y(F)$ is $F$ and by the Equation 4.1 we know;

$$\text{Hom}_{\pi(Y(F))}(y_1, y_2) = \{[u] \in \pi(X) \mid \text{such that } (F(p)[u])y_1 = y_2 \}.$$ 

This means, if $[u], [v] \in \text{Hom}_{\pi(Y(F))}(y_1, y_2)$ then

$$F([u])(\text{Hom}_{\pi(X)}(x_0, x_1)) = \text{Hom}_{\pi(X)}(x_0, x_2)$$

and

$$F([v])(\text{Hom}_{\pi(X)}(x_0, x_1)) = \text{Hom}_{\pi(X)}(x_0, x_2).$$

Then for some $[u], [w'] \in \text{Hom}_{\pi(X)}(x_0, x_1), [wu] \simeq [w'v]$ from two equalities of sets stated above. So, there is a homotopy between $wu$ and $w'u$. Also we know both $w$ and $w'$ has ending point $x_1$. So there is a homotopy between $u$ and $v$ which means these two paths are homotopic to each other. Therefore $Y(F)$ is null homotopic so it is universal cover of $X$. \qed

References


Department of Mathematics, Bilkent University, 06800 Bilkent, Ankara, Turkey
E-mail address: adnan.cakar@ug.bilkent.edu.tr